The Lagrangian-Hamiltonian Formalism for Higher Order Field Theories

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Abstract

We generalize the Lagrangian-Hamiltonian formalism of Skinner and Rusk to higher order field theories on fiber bundles. As a byproduct we solve the long standing problem of defining, in a coordinate free manner, a Hamiltonian formalism for higher order Lagrangian field theories. Namely, our formalism does only depend on the action functional and, therefore, unlike previously proposed ones, is free from any relevant ambiguity.

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Introduction

First order Lagrangian mechanics can be generalized to higher order Lagrangian field theory. Moreover, the latter has got a very elegant geometric (and homological) formulation (see, for instance, [1]) on which there is general consensus. On the other hand, it seems that the generalization of Hamiltonian mechanics of Lagrangian systems to higher order field theory presents some more problems. Several answers have been proposed (see, for instance, [2, 3, 4, 5, 6, 7, 8, 9, 10] and the references therein) to the question: is there any reasonable, higher order, field theoretic analogue of Hamiltonian mechanics? In our opinion, none of them is satisfactorily natural, especially because of the common emergence of ambiguities due to either the arbitrary choice of a coordinate system [2] or the choice of a Legendre transform [7, 8, 10]. Namely, the latter seems not to be uniquely definable, except in the case of first order Lagrangian field theories when a satisfactory Hamiltonian formulation can be presented in terms of multisymplectic geometry (see, for instance, [11] - see also [12] for a recent review, and the references therein).

Nevertheless, it is still desirable to have a Hamiltonian formulation of higher order Lagrangian field theories enjoying the same nice properties as Hamiltonian mechanics, which 1) is natural, i.e., is independent of the choice of any structure other than the action functional, 2) gives rise to first order equations of motion, 3) takes advantage of the (pre-)symplectic geometry of the phase space, 4) is a natural starting point for gauge reduction, 5) is a natural starting point for quantization. The relationship between the Euler-Lagrange equations and the Hamilton equations deserves a special mention. The Legendre transform maps injectively solutions of the former to solutions of the latter, but, generically, Hamilton equations are not equivalent to Euler-Lagrange ones [11]. However, the difference between the two is a pure gauge and, therefore, it is irrelevant from a physical point of view.

In this paper we achieve the goal of finding a natural (in the above mentioned sense), geometric, higher order, field theoretic analogue of Hamiltonian mechanics of Lagrangian systems in two steps: first, we find a higher order, field theoretic analogue of the Skinner and Rusk "mixed Lagrangian-Hamiltonian" formalism [13, 14, 15] (see also [16]), which is rather straightforward (see [17] for a different, finite dimensional approach, to the same problem) and, second, we show that the derived theory "projects to a smaller space" which is naturally interpreted as phase space. Local expressions of the field equations on the phase space are nothing but de Donder equations [2] and, therefore, are naturally interpreted as the higher order, field theoretic, coordinate free analogue of Hamilton equations. A central role is played in the paper by multisymplectic geometry in the form of partial differential (PD, in the following) Hamiltonian system theory, which has been developed in [18].

The paper is divided into nine sections. The first four sections contain reviews of the

main aspects of the geometry underlying the paper. They have been included in order to make the paper as self-consistent as possible. The next five sections contain most of the original results.

The first section summarizes the notations and conventions adopted throughout the paper. It also contains references to some differential geometric facts which are often used in the subsequent sections. Finally, in Section 1 we briefly review the Skinner-Rusk formalism [14]. Section 2 is a short review of the geometric theory of partial differential equations (PDEs) (see, for instance, [20]). Section 3 outlines the properties of the main geometric structure of jet spaces and PDEs, the Cartan distribution, and reviews the geometric formulation of the calculus of variations [1]. Section 4 reviews the theory of PD-Hamiltonian systems and their PD-Hamilton equations [18]. Moreover, it contains examples of morphisms of PDEs coming from such theory. These examples are presented here for the first time.

In Section 5 we present the higher order, field theoretic analogue of Skinner-Rusk mixed Lagrangian-Hamiltonian formalism for mechanics. In Section 5 we also discuss the relationship between the field equations in the Lagrangian-Hamiltonian formalism (now on, ELH equations) and the Euler-Lagrange equations. In Section 6 we discuss some natural transformations of the ELH equations. As a byproduct, we prove that they are independent of the choice of a Lagrangian density, in the class of those yielding the same Euler-Lagrange equations, up to isomorphisms. ELH equations are, therefore, as natural as possible. In Section 7 we present our proposal for a Hamiltonian, higher order, field theory. Since we don't use any additional structure other than the ELH equations and the order of a Lagrangian density, we judge our theory satisfactorily natural. Moreover, the associated field equations (HDW equations) are first order and, more specifically, of the PD-Hamilton kind. In Section 8 we study the relationship between the HDW equations and the Euler-Lagrange equations. As a byproduct, we derive a new (and, in our opinion, satisfactorily natural) definition of Legendre transform for higher order, Lagrangian field theories. It is a non-local morphism of the Euler-Lagrange equations into the HDW equations. Finally, in Section 9 we apply the theory to the KdV equation which can be derived from a second order variational principle.

1 Notations, Conventions and the Skinner-Rusk Formalism

In this section we collect notations and conventions about some general constructions in differential geometry that will be used in the following.

Let N be a smooth manifold. If $L \subset N$ is a submanifold, we denote by $i_L : L \hookrightarrow N$ the inclusion. We denote by $C^\infty(N)$ the $\mathbb{R}$–algebra of smooth, $\mathbb{R}$–valued functions on N. We will always understand a vector field X on N as a derivation $X : C^\infty(N) \longrightarrow C^\infty(N)$.

We denote by $\mathrm{D}(N)$ the $C^\infty(N)$–module of vector fields over N, by $\Lambda(M) = \bigoplus_k \Lambda^k(N)$ the graded $\mathbb{R}$–algebra of differential forms over N and by $d : \Lambda(N) \longrightarrow \Lambda(N)$ the de Rham differential. If $F : N_1 \longrightarrow N$ is a smooth map of manifolds, we denote by $F^* : \Lambda(N) \longrightarrow \Lambda(N_1)$ the pull–back via F. We will understand everywhere the wedge product $\wedge$ of differential forms, i.e., for $\omega, \omega_1 \in \Lambda(N)$, we will write $\omega\omega_1$ instead of $\omega \wedge \omega_1$, .

Let $\alpha : A \longrightarrow N$ be an affine bundle (for instance, a vector bundle) and $F : N_1 \longrightarrow N$ a smooth map of manifolds. Let $\mathscr{A}$ be the affine space of smooth sections of α. For $a \in \mathscr{A}$ and $x \in N$ we put, sometimes, $a_x := a(x)$. The affine bundle on N_1 induced by α via F will be denoted by $F^\circ(\alpha) : F^\circ(A) \longrightarrow N$:

$$\begin{tikzcd} F^\circ(A) \arrow[r] \arrow[d, "F^\circ(\alpha)"'] & A \arrow[d, "\alpha"] \\ N_1 \arrow[r, "F"] & N \end{tikzcd} ,$$

and the space of its sections by $F^\circ(\mathscr{A})$. For any section $a \in \mathscr{A}$ there exists a unique section, which we denote by $F^\circ(a) \in F^\circ(\mathscr{A})$, such that the diagram

$$\begin{tikzcd} F^\circ(A) \arrow[r] & A \\ N_1 \arrow[r, "F"] \arrow[u, "F^\circ(a)"] & N \arrow[u, "a"'] \end{tikzcd}$$

commutes. If $F : N_1 \longrightarrow N$ is the embedding of a submanifold, we also write $\bullet\,|_F$ (or, simply, $\bullet\,|_{N_1}$) for $F^\circ(\,\bullet\,)$, and refer to it as the restriction of "$\bullet$" to N_1 (via F), whatever the object "$\bullet$" is (an affine bundle, its total space, its space of sections or a section of it).

We will always understand the sum over repeated upper-lower (multi)indexes. Our notations about multiindexes are the following. We will use the capital letters I, J, K for multiindexes. Let n be a positive integer. A multiindex of length k is a $k$tuple of indexes $I = (i_1, \ldots, i_k)$, $i_1, \ldots, i_k \leq n$. We identify multiindexes differing only by the order of the entries. If I is a multiindex of lenght k, we put $|I| := k$. Let $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$ be multiindexes, and i an index. We denote by IJ (resp. Ii) the multiindex $(i_1, \ldots, i_k, j_1, \ldots, j_l)$ (resp. $(i_1, \ldots, i_k, i)$).

We conclude this section by briefly reviewing those aspects of the Skinner-Rusk formalism for mechanics [13, 14, 15] that survive in our generalization to higher order field theory.

Let Q be an m-dimensional smooth manifold and $q^1, \ldots, q^m$ coordinates on it. Let $L \in C^\infty(TQ)$ be a Lagrangian function. Consider the induced bundle $\tau_0^\dagger := \tau_Q^\circ(\tau_Q^*) : T^\dagger := \tau_Q^\circ(T^*Q) \longrightarrow TQ$ from the cotangent bundle $\tau_Q^* : T^*Q \longrightarrow Q$ to Q, via the

tangent bundle $\tau_Q : TQ \longrightarrow Q$. Let $q : T^\dagger \longrightarrow T^*Q$ be the canonical projection (see Diagram (1))

$$\begin{array}{ccc} T^\dagger & \xrightarrow{q} & T^*Q \\ {\scriptstyle \tau_0^\dagger}\big\downarrow & & \big\downarrow{\scriptstyle \tau_Q^*} \\ TQ & \xrightarrow{\tau_Q} & Q \end{array} \quad . \tag{1}$$

On $T^\dagger$ there is a canonical function $h \in C^\infty(T^\dagger)$ defined by $h(v,p) := p(v)$, $v \in T_qQ$, $p \in T_q^*Q$, $q \in Q$. Consider also the function $E_L := h - \tau_0^{\dagger *}(L) \in C^\infty(T^\dagger)$. E_L is locally given by $E_L := p_i\dot{q}^i - L$, where $\ldots, q^i, \ldots, \dot{q}^i, \ldots, p_i, \ldots$ are standard coordinates on $T^\dagger$. Finally, put $\omega := q^*(\omega_0) \in \Lambda^2(T^\dagger)$, $\omega_0 \in \Lambda^2(T^*Q)$ being the canonical symplectic form on T^*Q, which is locally given by $\omega_0 = dp_i dq^i$. ω is a presymplectic form on $T^\dagger$ whose kernel is made of vector fields over $T^\dagger$ which are vertical with respect to the projection q. In the following, denote by $\mathbb{I} \subset \mathbb{R}$ a generic open interval. For a curve $\gamma : \mathbb{I} \ni t \longmapsto \gamma(t) \in T^\dagger$, consider equations

$$i_{\dot{\gamma}}\gamma^\circ(\omega) + \gamma^\circ(dE_L) = 0, \tag{2}$$

where $\dot{\gamma} \in \gamma^\circ(\mathrm{D}(T^\dagger))$ is the tangent field to γ. Equations (2) read locally

$$\left\{ \begin{array}{l} \frac{d}{dt}q^i = \dot{q}^i \\ p_i = \frac{\partial L}{\partial \dot{q}^i} \\ \frac{d}{dt}p_i = \frac{\partial L}{\partial q^i} \end{array} \right. .$$

In particular, for any solution γ of Equations (2) as above, $\tau_Q \circ \tau_0^\dagger \circ \gamma : \mathbb{I} \longrightarrow Q$ is a solution of the Euler-Lagrange equations determined by L. Notice that solutions of Equations (2) can only take values in the submanifold $\mathscr{P} \subset T^\dagger$ defined as

$$\mathscr{P} := \{P \in T^\dagger : \text{there exists } \Xi \in T_PT^\dagger \text{ such that } i_\Xi\omega_P + (dE_L)_P = 0\},$$

and that $\mathscr{P}$ is nothing but the graph of the Legendre transform $FL : TQ \longrightarrow T^*Q$. Finally, consider $\mathscr{P}_0 := q(\mathscr{P}) \subset T^*Q$. If $\mathscr{P}_0 \subset T^*Q$ is a submanifold and $q : \mathscr{P} \longrightarrow \mathscr{P}_0$ a submersion with connected fibers, then there exists a (unique) function $H \in C^\infty(\mathscr{P}_0)$ such that $q^*(H) = E_L|_{\mathscr{P}}$. Thus, for a curve $\sigma : \mathbb{I} \ni t \longmapsto \sigma(t) \in \mathscr{P}_0$, we can consider equations

$$i_{\dot{\sigma}}\sigma^\circ(\omega_0) + \sigma^\circ(dH) = 0, \tag{3}$$

where $\dot{\sigma} \in \sigma^\circ(\mathrm{D}(T^\dagger))$ is the tangent field to σ. For any solution $\gamma : \mathbb{I} \longrightarrow Q$ of the Euler-Lagrange equations, $FL \circ \dot{\gamma} : \mathbb{I} \longrightarrow \mathscr{P}_0$ is a solution of Equations (3). If the map $q : \mathscr{P} \longrightarrow T^*Q$ has maximum rank (which happens iff the matrix $\|\partial^2 L/\partial\dot{q}^i\partial\dot{q}^j\|_i^j$ has maximum rank, i.e., FL is a local diffeomorphism), then $\mathscr{P}_0 \subset T^*Q$ is an open submanifold, H is a local function on T^*Q, and Equations (3) read locally

$$\left\{ \begin{array}{l} \frac{d}{dt}q^i = \frac{\partial H}{\partial p_i} \\ \frac{d}{dt}p_i = -\frac{\partial H}{\partial q^i} \end{array} \right. ,$$

which are Hamilton equations. In this case, for any solution $\sigma : \mathbb{I} \longrightarrow T^*Q$ of Equations (3), $\tau_Q^* \circ \sigma : \mathbb{I} \longrightarrow Q$ is a solution of the Euler-Lagrange equations.

2 Geometry of Differential Equations

In this section we recall basic facts about the geometric theory of PDEs. For more details see, for instance, [20].

Let $\pi : E \longrightarrow M$ be a fiber bundle, $\dim M = n$, $\dim E = m + n$. In the following we denote by $U \subset M$ a generic open subset. For $0 \leq l \leq k \leq \infty$, let $\pi_k : J^k\pi \longrightarrow M$ be the bundle of k-jets of local sections of π, and $\pi_{k,l} : J^k\pi \longrightarrow J^l\pi$ the canonical projection. For any local section $s : U \longrightarrow E$ of π, we denote by $j_k s : U \longrightarrow J^k\pi$ its kth jet prolongation. For $x \in U$, put $[s]_x^k := (j_k s)(x)$. Any system of adapted to π coordinates $(\ldots, x^i, \ldots, u^\alpha, \ldots)$ on an open subset V of E gives rise to a system of jet coordinates on $\pi_{k,0}^{-1}(V) \subset J^k\pi$ which we denote by $(\ldots, x^i, \ldots, u^\alpha{}_{|I}, \ldots)$ or simply $(\ldots, x^i, \ldots, u_I^\alpha, \ldots)$ if this does not lead to confusion, $|I| \leq k$, where we put $u_{\mathsf{O}}^\alpha := u^\alpha$, $\alpha = 1, \ldots, m$.

Now, let $k < \infty$, $\tau_0 : T_0 \longrightarrow J^k\pi$ be a vector bundle, and $(\ldots, x^i, \ldots, u_I^\alpha, \ldots, v^a, \ldots)$ adapted to τ_0, local coordinates on T_0. A (possibly non-linear) *differential operator of order $\leq k$ 'acting on local sections of π, with values in τ_0'* (in short *'from π to τ_0'*) is a section $\Phi : J^k\pi \longrightarrow T_0$ of τ_0.

Let $\pi' : E' \longrightarrow M$ be another fiber bundle and $\varphi : E \longrightarrow E'$ a morphism of bundles. For any local section $s : U \longrightarrow E$ of π, $\varphi \circ s : U \longrightarrow E'$ is a local section of π'. Therefore, for all $0 \leq k \leq \infty$, φ induces a morphism $j_k\varphi : J^k\pi \longrightarrow J^k\pi'$ of the bundles π_k and π'_k defined by $(j_k\varphi)[s]_x^k := [\varphi \circ s]_x^k$, $x \in U$. Diagram

$$\begin{array}{ccc} J^l\pi & \xrightarrow{j_l\varphi} & J^l\pi' \\ {\scriptstyle \pi_{l,k}}\downarrow & & \downarrow{\scriptstyle \pi'_{l,k}} \\ J^k\pi & \xrightarrow{j_k\varphi} & J^k\pi' \end{array}$$

commutes for all $0 \leq k \leq l \leq \infty$. $j_k\varphi$ is called the *kth prolongation of φ.*

The above construction generalizes to differential operators as shown, for instance, in [22]. If Φ is a differential operator as above, we denote by $\Phi^{(l)}$ its lth prolongation. Moreover, put $\mathscr{E}_\Phi := \{\theta \in J^k\pi : \Phi(\theta) = 0\}$. $\mathscr{E}_\Phi$ is called the *(system of) PDE(s)* determined by Φ. For $0 \leq l \leq \infty$ put also $\mathscr{E}_\Phi^{(l)} := \mathscr{E}_{\Phi^{(l)}} \subset J^{k+l}\pi$. If $\mathscr{E}_\Phi$ is locally defined by

$$\Phi^a(\ldots, x^i, \ldots, u_I^\alpha, \ldots) = 0, \quad a = 1, \ldots, p \tag{4}$$

$\ldots, \Phi^a := \Phi^*(v^a), \ldots$ being local functions on $J^k\pi$, then $\mathscr{E}_\Phi^{(l)}$ is locally defined by

$$(D_J\Phi^a)(\ldots, x^i, \ldots, u_I^\alpha, \ldots) = 0, \quad a = 1, \ldots, p, \ |J| \leq l, \tag{5}$$

where $D_{(j_1,\dots j_l)} := D_{j_1} \circ \cdots \circ D_{j_l}$, and $D_j := \partial/\partial x^j + u^\alpha_{Ij}\partial/\partial u^\alpha_I$ is the *jth total derivative,* $j, j_1, \dots, j_l = 1, \dots, m$. $\mathscr{E}^{(l)}_\Phi$ is called the lth prolongation of the PDE $\mathscr{E}_\Phi$. In the following we put $\partial^I_\alpha := \partial/\partial u^\alpha_I$, $\alpha = 1, \dots, m$.

A local section s of π is a *(local) solution of* $\mathscr{E}_\Phi$ iff, by definition, $\operatorname{im} j_k s \subset \mathscr{E}_\Phi$ or, which is the same, $\operatorname{im} j_{k+l} s \subset \mathscr{E}^{(l)}_\Phi$ for some $l \leq \infty$. Notice that the ∞th prolongation of $\mathscr{E}_\Phi$, $\mathscr{E}^{(\infty)}_\Phi \subset J^\infty\pi$, is an inverse limit of the sequence of maps

$$M \xleftarrow{\ \pi_k\ } \mathscr{E}_\Phi \longleftarrow \cdots \xleftarrow{\pi_{k+l,k+l-1}} \mathscr{E}^{(l)}_\Phi \xleftarrow{\pi_{k+l+1,k+l}} \mathscr{E}^{(l+1)}_\Phi \longleftarrow \cdots \tag{6}$$

and consists of "formal solutions" of $\mathscr{E}_\Phi$, i.e., possibly non-converging Taylor series fulfilling (5) for every l.

$J^\infty\pi$ is not a finite dimensional smooth manifold. However, it is a *pro-finite dimensional smooth manifold.* For an introduction to the geometry of pro-finite dimensional smooth manifolds see [21] (see also [22], and [23, 24] for different approaches). In the following we will only consider *regular* PDEs, i.e., PDEs $\mathscr{E}_\Phi$ such that $\mathscr{E}^{(\infty)}_\Phi \subset J^\infty\pi$ is a smooth pro-finite dimensional submanifold in $J^\infty\pi$, i.e., $\pi_{\infty,l}(\mathscr{E}^{(\infty)}_\Phi) \subset J^l\pi$ is a smooth submanifold and $\pi_{l+1,l} : \pi_{\infty,l+1}(\mathscr{E}^{(\infty)}_\Phi) \longrightarrow \pi_{\infty,l}(\mathscr{E}^{(\infty)}_\Phi)$ is a smooth bundle for all $l \geq 0$.

There is a dual concept to the one of a pro-finite dimensional manifold, i.e., the concept of a *filtered smooth manifold* which will be used in the following. We do not give here a complete definition of a filtered manifold, which would take too much space. Rather, we will just outline it. Basically, a filtered smooth manifold is a(n equivalence class of) set(s) $\mathscr{O}$ together with a sequence of embeddings of closed submanifolds

$$\mathscr{O}_0 \xhookrightarrow{\ i_{0,1}\ } \mathscr{O}_1 \xhookrightarrow{\ i_{0,1}\ } \cdots \hookrightarrow \mathscr{O}_{k-1} \xhookrightarrow{\ i_{k-1,k}\ } \mathscr{O}_k \xhookrightarrow{\ i_{k,k+1}\ } \cdots \tag{7}$$

and inclusions $i_k : \mathscr{O}_k \hookrightarrow \mathscr{O}$, $k \geq 0$, such that $\mathscr{O}$ (together with the i_k's) is a direct limit of (7). The tower of algebra epimorphisms

$$C^\infty(\mathscr{O}_0) \longleftarrow \cdots \xleftarrow{i^*_{k-1,k}} C^\infty(\mathscr{O}_k) \xleftarrow{i^*_{k,k+1}} C^\infty(\mathscr{O}_{k+1}) \longleftarrow \cdots \tag{8}$$

is associated to sequence (7). We define $C^\infty(\mathscr{O})$ to be the inverse limit of the tower (8). Every element in $C^\infty(\mathscr{O})$ is naturally a function on $\mathscr{O}$. Thus, we interpret $C^\infty(\mathscr{O})$ as the algebra of *smooth functions on $\mathscr{O}$.* Clearly, there are canonical "restriction homomorphisms" $i^*_k : C^\infty(\mathscr{O}) \longrightarrow C^\infty(\mathscr{O}_k)$, $k \geq 0$. Differential calculus over $\mathscr{O}$ may then be introduced as *differential calculus over* $C^\infty(\mathscr{O})$ [21] respecting the sequence (8). Since the main constructions (smooth maps, vector fields, differential forms, jets and differential operators, etc.) of such calculus and their properties do not look very different from the analogous ones in finite-dimensional differential geometry we will not insist on this. Just as an instance, we report here the definition of a differential

form ω on $\mathscr{O}$: it is just a sequence of differential forms $\omega_k \in \Lambda(\mathscr{O}_k)$, $k \geq 0$, such that $i^*_{k-1,k}(\omega_k) = \omega_{k-1}$ for all k.

Finally, notice that, allowing for the $\mathscr{O}_k$'s in (7) to be pro-finite dimensional manifolds, we obtain a more general object than both a pro-finite dimensional and a filtered manifold. We will generically refer to such an object as an *infinite dimensional smooth manifold* or even just a *smooth manifold* if this does not lead to confusion. Our main example of such a kind of infinite dimensional manifold will be presented in the beginning of Section 5.

3 The Cartan Distribution and the Lagrangian Formalism

Let $\pi : E \longrightarrow M$ and Φ be as in the previous section. In the following we will simply write J^k for $J^k\pi$, $k \leq \infty$, and $\mathscr{E}$ for $\mathscr{E}_\Phi^{(\infty)}$. $\mathscr{E}$ will be referred to simply as a PDE (imposed on sections of π) if this does not lead to confusion. Notice that for $\Phi = 0$, $\mathscr{E} = \mathscr{E}_\Phi^{(\infty)} = J^\infty$.

Recall that J^∞ is canonically endowed with the Cartan distribution [20]

$$\mathscr{C} : J^\infty \ni \theta \longmapsto \mathscr{C}_\theta \subset T_\theta J^\infty$$

which is locally spanned by total derivatives, D_i, $i = 1, \ldots, n$. $\mathscr{C}$ is a flat connection in π_∞ which we call the *Cartan connection*. Moreover, it restricts to $\mathscr{E}$ in the sense that $\mathscr{C}_\theta \subset T_\theta \mathscr{E}$ for any $\theta \in \mathscr{E}$. Therefore, the (infinite prolongation of) any PDE is naturally endowed with an involutive distribution whose n-dimensional integral submanifolds are of the form $j_\infty s$, with $s : U \longrightarrow E$ a (local) solution of $\mathscr{E}_\Phi$. In the following we will identify the space of n-dimensional integral submanifolds of $\mathscr{C}$ and the space of local solutions of $\mathscr{E}_\Phi$.

Let $\pi' : E' \longrightarrow M$ be another bundle and $\mathscr{E}' \subset J^\infty\pi'$ (the infinite prolongation of) a PDE imposed on sections of π'. A smooth map $F : \mathscr{E}' \longrightarrow \mathscr{E}$ is called a *morphism of PDEs* iff it respects the Cartan distributions, i.e., $(d_{\theta'}F)(\mathscr{C}_{\theta'}) = \mathscr{C}_{F(\theta')}$ for any $\theta' \in \mathscr{E}'$. The idea of *non-local variables* in the theory of PDEs can be formalized geometrically by special morphisms of PDEs called *coverings* [25] (see also [26]). A *covering* is a morphism $\psi : \widehat{\mathscr{E}} \longrightarrow \mathscr{E}$ of PDEs which is surjective and submersive. A covering $\psi : \widehat{\mathscr{E}} \longrightarrow \mathscr{E}$ clearly sends local solutions of $\widehat{\mathscr{E}}$ to local solutions of $\mathscr{E}$. If there exists a covering $\psi : \widehat{\mathscr{E}} \longrightarrow \mathscr{E}$ of PDEs we also say that the PDE $\widehat{\mathscr{E}}$ *covers the PDE* $\mathscr{E}$ *(via* ψ*)*. Fiber coordinates on the total space $\widehat{\mathscr{E}}$ of a covering $\psi : \widehat{\mathscr{E}} \longrightarrow \mathscr{E}$ are naturally interpreted as non-local variables on $\mathscr{E}$. Also notice that given a solution s of the PDE $\mathscr{E}$, a covering $\psi : \widehat{\mathscr{E}} \longrightarrow \mathscr{E}$ determines a whole family of solutions of $\widehat{\mathscr{E}}$ "projecting onto s via ψ", so that ψ may be interpreted, to some extent, as a fibration over the

space of solutions of $\mathscr{E}$. Many relevant constructions in the theory of PDEs (including Lax pairs, Bäcklund transformations, etc.) are duly formalized in geometrical terms by using coverings.

The Cartan distribution and the fibered structure $\pi_\infty : J^\infty \longrightarrow M$ of J^∞ determine a splitting of the tangent bundle $TJ^\infty \longrightarrow J^\infty$ into the Cartan or horizontal part $\mathscr{C}$ and the vertical (with respect to π_∞) part. Accordingly, the de Rham complex of J^∞, $(\Lambda(J^\infty), d)$, splits in the *variational bi-complex* $(\mathscr{C}^\bullet\Lambda \otimes \overline{\Lambda}, \overline{d}, d^V)$, (here and in what follows tensor products will be always over $C^\infty(J^\infty)$ if not otherwise specified), where $\mathscr{C}^\bullet\Lambda$ and $\overline{\Lambda}^\bullet$ are the algebras of *Cartan forms* and *horizontal forms* respectively. d^V and $\overline{d}$ are the *vertical* and the *horizontal de Rham differential*, respectively (see, for instance, [20] for details). The variational bicomplex allows a cohomological formulation of the calculus of variations [1, 20, 21, 19]. In the second part of this section we briefly review it.

In the following we will understand isomorphism $\Lambda(J^\infty) \simeq \mathscr{C}^\bullet\Lambda \otimes \overline{\Lambda}$. The complex

$$0 \longrightarrow C^\infty(J^\infty) \xrightarrow{\overline{d}} \overline{\Lambda}^1 \xrightarrow{\overline{d}} \cdots \longrightarrow \overline{\Lambda}^q \xrightarrow{\overline{d}} \overline{\Lambda}^{q+1} \xrightarrow{\overline{d}} \cdots$$

is called the *horizontal de Rham complex*. An element $\mathscr{L} \in \overline{\Lambda}^n$ is naturally interpreted as a *Lagrangian density* and its cohomology class $[\mathscr{L}] \in \overline{H}^n := H^n(\overline{\Lambda}, \overline{d})$ as an *action functional* on sections of π. The associated Euler-Lagrange equations can then be obtained as follows.

Consider the complex

$$0 \longrightarrow \mathscr{C}\Lambda^1 \xrightarrow{\overline{d}} \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^1 \xrightarrow{\overline{d}} \cdots \longrightarrow \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^q \xrightarrow{\overline{d}} \cdots, \tag{9}$$

and the $C^\infty(J^\infty)$-submodule $\varkappa^\dagger \subset \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n$ generated by elements in $\mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n \cap \Lambda(J^1\pi)$. $\varkappa^\dagger$ is locally spanned by elements $(du^\alpha - u^\alpha_i dx^i) \otimes d^n x$, where we put $d^n x := dx^1 \cdots dx^n$.

Theorem 1 *[1] Complex (9) is acyclic in the qth term, for $q \neq n$. Moreover, for any $\omega \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n$ there exists a unique element $\boldsymbol{E}_\omega \in \varkappa^\dagger \subset \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n$ such that $\boldsymbol{E}_\omega - \omega = \overline{d}\vartheta$ for some $\vartheta \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$ and the correspondence $H^n(\mathscr{C}\Lambda^1 \otimes \overline{\Lambda}, \overline{d}) \ni [\omega] \longmapsto \boldsymbol{E}_\omega \in \varkappa^\dagger$ is a vector space isomorphism. In particular, for $\omega = d^V\mathscr{L}$, $\mathscr{L} \in \overline{\Lambda}^n$ being a Lagrangian density locally given by $\mathscr{L} = L d^n x$, L a local function on $C^\infty(J^\infty)$, $\boldsymbol{E}(\mathscr{L}) := \boldsymbol{E}_\omega$ is locally given by $\boldsymbol{E}(\mathscr{L}) = \frac{\delta L}{\delta u^\alpha}(du^\alpha - u^\alpha_i dx^i) \otimes d^n x$ where $\frac{\delta L}{\delta u^\alpha} := (-)^{|I|} D_I \partial^I_\alpha L$ are the Euler-Lagrange derivatives of L.*

In view of the above theorem, $\boldsymbol{E}(\mathscr{L})$ does not depend on the choice of $\mathscr{L}$ in a cohomology class $[\mathscr{L}] \in \overline{H}^n$ and it is naturally interpreted as the left hand side of the Euler-Lagrange (EL) equations determined by $\mathscr{L}$. In the following we will denote by

$\mathscr{E}_{EL} \subset J^\infty$ the (infinite prolongation of the) EL equations determined by a Lagrangian density. Any $\vartheta \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$ such that

$$\boldsymbol{E}(\mathscr{L}) - d^V\mathscr{L} = \overline{d}\vartheta \tag{10}$$

will be called a *Legendre form* [10]. Equation (10) may be interpreted as the *first variation formula* for the Lagrangian density $\mathscr{L}$. In this respect, the existence of a global Legendre form was first discussed in [27].

Remark 1 *Notice that, if $\vartheta \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$ is a Legendre form for a Lagrangian density $\mathscr{L} \in \overline{\Lambda}^n$, then $\vartheta + d^V\varrho$ is a Legendre form for the $\overline{d}$-cohomologous Lagrangian density $\mathscr{L} + \overline{d}\varrho$, $\varrho \in \overline{\Lambda}^{n-1}$, which determines the same EL equations as $\mathscr{L}$. Moreover, any two Legendre forms ϑ, ϑ' for the same Lagrangian density differ by a $\overline{d}$-closed, and, therefore, $\overline{d}$-exact form, i.e., $\vartheta - \vartheta' = \overline{d}\lambda$, for some $\lambda \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-2}$.*

Remark 2 *Finally, notice that complex (9) restricts to* holonomic sections *$j_\infty s$ of π_∞, s being a local sections of π, in the sense that, for any such s, there is a (unique) complex*

$$0 \longrightarrow \mathscr{C}\Lambda^1|_j \xrightarrow{\overline{d}|_j} \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^1|_j \xrightarrow{\overline{d}|_j} \cdots \longrightarrow \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^q|_j \xrightarrow{\overline{d}|_j} \cdots, \tag{11}$$

where $j := j_\infty s$, such that the restriction map $\mathscr{C}\Lambda^1 \otimes \overline{\Lambda} \longrightarrow \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}|_j \simeq \mathscr{C}\Lambda^1|_j \otimes_{C^\infty(M)} \Lambda(M)$ is a morphism of complexes. Moreover, complex (11) is acyclic in the qth term and the correspondence defined by $H^n(\mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n|_j, \overline{d}|_j) \ni [\omega|_j] \longmapsto \boldsymbol{E}_\omega|_j \in \varkappa^\dagger|_j$, $\omega \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n$, is a vector space isomorphism.

4 Partial Differential Hamiltonian Systems

In [18] we defined a PD analogue of the concept of Hamiltonian system on an abstract symplectic manifold which we called a *PD-Hamiltonian system*. In this section we briefly review those definitions and results in [18] which we will need in the following.

Let $\alpha : P \longrightarrow M$ be a fiber bundle, $A := C^\infty(P)$, $x^1, \ldots, x^n$ coordinates on M, $\dim M = n$, and $q^1, \ldots, q^m$ fiber coordinates on P, $\dim P = n + m$. Denote by $C(P, \alpha)$ the space of (Ehresmann) connections in α. $C(P, \alpha)$ identifies canonically with the space of sections of the first jet bundle $\alpha_{1,0} : J^1\alpha \longrightarrow P$ and in the following we will understand such identification. In particular, for $\nabla \in C(P, \alpha)$, we put $\ldots, \nabla_i^A := \nabla^*(q_i^A), \ldots$, $\ldots, q_i^A, \ldots$ being jet coordinates on $J^1\alpha$.

Denote by $\Lambda_1 = \bigoplus_k \Lambda_1^k \subset \Lambda(P)$ the differential (graded) ideal in $\Lambda(P)$ made of differential forms on P vanishing when pulled-back to fibers of α, by $\Lambda_p = \bigoplus_k \Lambda_p^k$ its p-th exterior power, $p \geq 0$, and by $V\Lambda(P, \alpha) = \bigoplus_k V\Lambda^k(P, \alpha)$ the quotient differential algebra $\Lambda(P)/\Lambda_1$, $d^V : V\Lambda(P, \alpha) \longrightarrow V\Lambda(P, \alpha)$ being its (quotient) differential.

Remark 3 *For instance, if $\alpha = \pi_\infty : P = J^\infty \longrightarrow M$, then, using the Cartan connection $\mathscr{C} \in C(J^\infty, \pi_\infty)$, one can canonically identify $V\Lambda^1(J^\infty, \pi_\infty)$ with $\mathscr{C}\Lambda^1$ and d^V with the vertical de Rham differential. More generally, for any $k \geq 0$, $V\Lambda^1(J^k, \pi_k) \otimes_{C^\infty(J^k\pi)} C^\infty(J^{k+1}\pi)$ identifies canonically with the $C^\infty(J^{k+1}\pi)$-module $\mathscr{C}\Lambda^1 \cap \Lambda(J^{k+1}\pi)$ of $(k+1)$th order* Cartan forms.

Now, for any $k \geq 0$, put $\Omega^k(P, \alpha) := \Lambda^{k+n-1}_{n-1}$ and $\underline{\Omega}^k(P, \alpha) := \Omega^k(P, \alpha)/\Lambda^{k+n-1}_n$. It is easy to show that $\underline{\Omega}^k(P, \alpha) \simeq V\Lambda^k(P, \alpha) \otimes_A \Lambda^{n-1}_{n-1}$. An element $\omega \in \Omega^k(P, \alpha)$ determines an affine map

$$C(P, \alpha) \ni \nabla \longmapsto i_\nabla \omega := \mathfrak{p}_\nabla(\omega) \in V\Lambda^{k-1}(P, \alpha) \otimes_A \Lambda^n_n, \tag{12}$$

where

$$\mathfrak{p}_\nabla : \Lambda(P) \longrightarrow V\Lambda^{k-1}(P, \alpha) \otimes_A \Lambda^n_n$$

is the canonical projection determined by the connection ∇. The linear part of the affine map (12) naturally identify with the class $\omega + \Lambda^{k+n-1}_n$ in $\underline{\Omega}^k(P, \alpha)$ (see [18] for details). Notice that, since (12) is affine, it is actually point-wise and, therefore, can be restricted to maps. Namely, if $F : P_1 \longrightarrow P$ is a smooth map, $\square \in F^\circ(C(P, \alpha))$, then an element $i_\square F^\circ(\omega) \in F^\circ(V\Lambda^{k-1}(P, \alpha) \otimes_A \Lambda^n_n)$ is defined in an obvious way.

Definition 1 *A* PD-Hamiltonian system *on the fiber bundle $\alpha : P \longrightarrow M$ is an element $\omega \in \Omega^2(P, \alpha)$ such that $d\omega = 0$. The first order PDEs*

$$i_{j_1\sigma}\omega|_\sigma = 0$$

on (local) sections σ of α are called the PD-Hamilton equations determined by ω. *Geometrically, they correspond to the submanifold*

$$\mathscr{E}^{(0)}_\omega := \{\theta \in J^1\alpha : i_\theta \omega_p = 0,\ p = \alpha_{1,0}(\theta)\} \subset J^1\alpha.$$

Let ω be a PD-Hamiltonian system on the bundle $\alpha : P \longrightarrow M$ and consider the subset $P_1 := \alpha_{1,0}(\mathscr{E}^{(0)}_\omega) \subset P$. In the following we will assume $P_1 \subset P$ to be a submanifold and $\alpha_1 := \alpha|_{P_1} : P_1 \longrightarrow M$ to be a subbundle of α. α_1 is called *the first constraint subbundle of ω*.

As an example, consider the following canonical constructions. Let $\alpha : P \longrightarrow M$ be a fiber bundle and $\ldots, q^A, \ldots$ fiber coordinates on P. $\Omega^1(P, \alpha)$ (resp. $\underline{\Omega}^1(P, \alpha)$) is the $C^\infty(P)$-module of sections of a vector bundle $\mu_0\alpha : \mathscr{M}\alpha \longrightarrow P$ (resp. $\tau^\dagger_0\alpha : J^\dagger\alpha \longrightarrow P$), called the *multimomentum bundle of α* (resp. the *reduced multimomentum bundle of α*). Recall that there is a tautological element $\Theta_\alpha \in \Omega^1(\mathscr{M}\alpha, \mu\alpha)$ (resp. $\underline{\Theta}_\alpha \in \underline{\Omega}^1(J^\dagger\alpha, \tau^\dagger\alpha)$), where $\mu\alpha := \alpha \circ \mu_0\alpha$ (resp. $\tau^\dagger\alpha := \alpha \circ \tau^\dagger_0\alpha$), which in standard

coordinates $\ldots, x^i, \ldots, q^A, \ldots, p^i_A, \ldots, p$ on $\mathscr{M}\alpha$ (resp. $\ldots, x^i, \ldots, q^A, \ldots, p^i_A, \ldots$ on $J^\dagger\alpha$) is given by

$$\Theta_\alpha = p^i_A dq^A d^{n-1}x_i - p d^n x \quad (\text{resp. } \underline{\Theta}_\alpha = p^i_A d^V q^A \otimes d^{n-1}x_i),$$

where $d^{n-1}x_i := i_{\partial/\partial x^i} d^n x$ [28]. $d\Theta_\alpha$ is a PD-Hamiltonian system on $\mu\alpha$ locally given by

$$d\Theta_\alpha = dp^i_A dq^A d^{n-1}x_i - dp d^n x.$$

Notice that $d\Theta_\alpha$ determines empty PD-Hamilton equations.

Example 1 *A PD-Hamiltonian system is canonically determined, on the fiber bundle $\alpha : P \longrightarrow M$, by the following data: a connection ∇ in α and a differential form $\mathscr{L} \in \Lambda^n_n$. Let $\ldots, q^A, \ldots$ be fiber coordinates in P and $\ldots, x^i, \ldots, q^A, \ldots, p^i_A, \ldots, p$ (resp. $\ldots, x^i, \ldots, q^A, \ldots, p^i_A, \ldots$) standard coordinates in $\mathscr{M}\alpha$ (resp. $J^\dagger\alpha$). Let $\mathscr{L}$ be locally given by $\mathscr{L} = L d^n x$, L a local function on P. Obviously, ∇ determines a section $\Sigma_\nabla : J^\dagger\alpha \longrightarrow \mathscr{M}\alpha$ of the projection $\mathscr{M}\alpha \longrightarrow J^\dagger\alpha$, which in local standard coordinates reads $\Sigma^*_\nabla(p) = p^i_A \nabla^A_i$. Put $\Theta_\nabla := \Sigma^*_\nabla(\Theta_\alpha)$. In local standard coordinates, $\Theta_\nabla = p^i_A dq^A d^{n-1}x_i - p^i_A \nabla^A_i d^n x$. Put also,*

$$\Theta_{\mathscr{L},\nabla} := \Theta_\nabla + (\tau^\dagger_0\alpha)^*(\mathscr{L}).$$

Locally, $\Theta_{\mathscr{L},\nabla} = p^i_A dq^A d^{n-1}x_i - E_{\mathscr{L},\nabla} d^n x$, where $E_{\mathscr{L},\nabla} := p^i_A \nabla^A_i - L$. Finally, consider $\omega_{\mathscr{L},\nabla} := d\Theta_{\mathscr{L},\nabla}$. Locally,

$$\omega_{\mathscr{L},\nabla} = dp^i_A dq^A d^{n-1}x_i - dE_{\mathscr{L},\nabla} d^n x.$$

$\omega_{\mathscr{L},\nabla}$ is the PD-Hamiltonian system on $\tau^\dagger\alpha$ determined by ∇ and $\mathscr{L}$. *The associated PD-Hamilton equations read locally*

$$\left\{ \begin{array}{l} p^i_{A,i} = \frac{\partial}{\partial q^A} L - p^i_B \frac{\partial}{\partial q^A} \nabla^B_i \\ q^A{}_{,i} = \nabla^A_i \end{array} \right. ,$$

where we denoted by "$\bullet_{,i}$" the partial derivative of "$\bullet$" with respect to the ith independent variable x^i, $i = 1, \ldots, n$.

We conclude this section by discussing two examples of morphisms of PDEs coming from the theory of PD-Hamiltonian systems.

Example 2 *Let $\alpha : P \longrightarrow M$ be a fiber bundle, ω a PD-Hamiltonian system on it, $\alpha' : P' \longrightarrow M$ another fiber bundle, $\beta : P' \longrightarrow P$ a surjective, submersive, fiber bundle morphism, and $\omega' := \beta^*(\omega)$. ω' is a PD-Hamiltonian system on α'. Denote by $\mathscr{E} \subset J^\infty\alpha$ (resp. $\mathscr{E}' \subset J^\infty\alpha'$) the ∞th prolongation of the PD-Hamilton equations determined by ω (resp. ω'). We want to compare $\mathscr{E}$ and $\mathscr{E}'$. In order to do this, notice, preliminarily, that $J^\infty\alpha'$ covers $J^\infty\alpha$ via $j_\infty\beta : J^\infty\alpha' \longrightarrow J^\infty\alpha$. Moreover, it can be easily checked that a local section σ' of α' is a solution of $\mathscr{E}'$ iff the section $\beta \circ \sigma'$ of α is a solution of $\mathscr{E}$. We now prove the formal version of this fact.*

Proposition 2 *$(j_\infty\beta)(\mathscr{E}') \subset \mathscr{E}$ and $j_\infty\beta : \mathscr{E}' \longrightarrow \mathscr{E}$ is a covering.*

Proof. *Consider $j_1\beta : J^1\alpha' \longrightarrow J^1\alpha$. It is easy to check that $\mathscr{E}^{(0)}_{\omega'} = (j_1\beta)^{-1}(\mathscr{E}^{(0)}_{\omega}) \subset J^1\alpha'$. Similarly, $\mathscr{E}' = (j_\infty\beta)^{-1}(\mathscr{E}) \subset J^\infty\alpha'$. In particular, $j_\infty\beta : \mathscr{E}' \longrightarrow \mathscr{E}$ is the "restriction" of $j_\infty\beta : J^\infty\alpha' \longrightarrow J^\infty\alpha$ to $\mathscr{E} \subset J^\infty\alpha$ and, therefore, is a covering.* ■

Example 3 *Let $\alpha : P \longrightarrow M$, ω and $\mathscr{E} \subset J^\infty\alpha$ be as in the above example, and $\alpha_1 : P_1 \longrightarrow M$ the first constraint subbundle of ω. Put $\omega_1 := i^*_{P_1}(\omega)$. ω_1 is a PD-Hamiltonian system on α_1. Denote by $\mathscr{E}_1 \subset J^\infty\alpha_1$ the ∞th prolongation of the PD-Hamilton equations determined by ω_1. We want to compare $\mathscr{E}$ and $\mathscr{E}_1$. In order to do this, notice, preliminarily, that $J^\infty\alpha_1$ may be understood as a submanifold in $J^\infty\alpha$ via $j_\infty i_{P_1} : J^\infty\alpha_1 \hookrightarrow J^\infty\alpha$. Moreover, it can be easily checked that any solution of $\mathscr{E}$ is also a solution of $\mathscr{E}_1$ (while the vice-versa is generically untrue). We now prove the formal version of this fact.*

Proposition 3 *$\mathscr{E} \subset \mathscr{E}_1$.*

Proof. *Recall that the projection $\alpha_{1,0} : J^1\alpha \longrightarrow P$ sends $\mathscr{E}^{(0)}_\omega$ to P_1. As a consequence, $\mathscr{E} \subset J^\infty\alpha_1$. Moreover, by definition of ∞th prolongation of a PDE, it is easy to check that*

$$
\begin{aligned}
\mathscr{E} &= \mathscr{E} \cap J^\infty\alpha_1 \\
&= \{\theta = [\sigma]^\infty_x \in J^\infty\alpha_1 : \ [i_{j_1\sigma}\omega|_\sigma]^\infty_x = 0,\ x \in M\} \\
&\subset \{\theta = [\sigma]^\infty_x \in J^\infty\alpha_1 : \ [i_{j_1\sigma}\omega_1|_\sigma]^\infty_x = 0,\ x \in M\} \\
&= \mathscr{E}_1.
\end{aligned}
$$

■

5 Lagrangian-Hamiltonian Formalism

In this section we show that the Skinner-Rusk mixed Lagrangian-Hamiltonian formalism for first order mechanics [13, 14, 15] (see Section 1) is straightforwardly generalized to higher order Lagrangian field theories.

First of all, let us present our main example of a filtered manifold. Let $\pi : E \longrightarrow M$ be a fiber bundle. Consider the infinite jet bundle $\pi_\infty : J^\infty \longrightarrow M$ for which $\Lambda^q_q = \overline{\Lambda}^q$, $q \geq 0$. Moreover, the $C^\infty(J^\infty)$-module $\underline{\Omega}^1(J^\infty, \pi_\infty) \simeq \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$ is canonically filtered by vector subspaces $W_k := \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1} \cap \Lambda(J^{k+1}\pi)$, $k \geq 0$. Denote by $\underline{\Omega}^1_k \subset \underline{\Omega}^1(J^\infty, \pi_\infty)$ the $C^\infty(J^\infty)$-submodule generated by W_k, $k \geq 0$. Then, for all k, $\underline{\Omega}^1_k$ is canonically isomorphic to $C^\infty(J^\infty) \otimes_{C^\infty(J^{k+1})} W_k$ and

$$
\underline{\Omega}^1_0 \subset \underline{\Omega}^1_1 \subset \cdots \subset \underline{\Omega}^1_k \subset \underline{\Omega}^1_{k+1} \subset \cdots \subset \underline{\Omega}^1(J^\infty, \pi_\infty), \tag{13}
$$

is a sequence of $C^\infty(J^\infty)$-submodules. Notice that, for any k, $\underline{\Omega}^1_k$ is the module of sections of a finite-dimensional vector bundle $J^\dagger_k \longrightarrow J^\infty$. Moreover, the inclusions (13) determine inclusions

$$J^\dagger_0 \subset J^\dagger_1 \subset \cdots \subset J^\dagger_k \subset J^\dagger_{k+1} \subset \cdots$$

of vector bundles. $J^\dagger := \bigcup_k J^\dagger_k$ is then an infinite dimensional (filtered) manifold and the canonical projection $\tau^\dagger_0 : J^\dagger \longrightarrow J^\infty$ an *infinite dimensional vector bundle over* J^∞ whose module of sections identifies naturally with $\underline{\Omega}^1(J^\infty, \pi_\infty)$. We conclude that $\tau^\dagger_0 : J^\dagger \longrightarrow J^\infty$ is naturally interpreted as the reduced multimomentum bundle of π_∞. Denote by $\ldots, x^i, \ldots, u^\alpha_I, \ldots, p^{I.i}_\alpha, \ldots$ standard coordinates on $J^\dagger$. We will also consider the bundle structures $J^\dagger_k \longrightarrow M$, $k \geq 0$, and $\tau^\dagger := \pi_\infty \circ \tau^\dagger_0 : J^\dagger \longrightarrow M$.

In the following we denote by $U' \subset J^\infty$ a generic open subset. Notice that any (local) element $\vartheta \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1} = \underline{\Omega}^1(J^\infty, \pi_\infty)$, in particular a (local) Legendre form, is naturally interpreted as a section $\vartheta : U' \longrightarrow J^\dagger$ of $\tau^\dagger_0$. Put then $\ldots, \vartheta^{I.i}_\alpha := \vartheta^*(p^{I.i}_\alpha), \ldots$ which are local functions on J^∞ such that $\vartheta = \vartheta^{I.i}_\alpha(du^\alpha_I - u^\alpha_{Ii}dx^i) \otimes d^{n-1}x_i$. It follows that, locally,

$$\overline{d}\vartheta = -(D_i\vartheta^{I.i}_\alpha + \delta^I_{Ji}\vartheta^{J.i}_\alpha)(du^\alpha_I - u^\alpha_{Ii}dx^i) \otimes d^n x \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^n,$$

where $\delta^I_K = 0$ if $I \neq K$, while $\delta^I_K = 1$ if $I = K$.

Now, in Example 1, put $\alpha = \pi_\infty : P = J^\infty \longrightarrow M$ and $\nabla = \mathscr{C}$, the Cartan connection in π_∞. $\mathscr{L} \in \Lambda^n_n = \overline{\Lambda}^n$ is then a Lagrangian density in π. Put $\Sigma := \Sigma_{\mathscr{C}}$, $\Theta_{\mathscr{L}} := \Theta_{\mathscr{L},\mathscr{C}}$ and $\omega_{\mathscr{L}} := \omega_{\mathscr{L},\mathscr{C}}$. $\omega_{\mathscr{L}}$ is a PD-Hamiltonian system on $\tau^\dagger : J^\dagger \longrightarrow M$ canonically determined by $\mathscr{L}$. Locally,

$$\omega_{\mathscr{L}} = dp^{I.i}_\alpha du^\alpha_I d^{n-1}x_i - dE_{\mathscr{L}} d^n x,$$

where $E_{\mathscr{L}} := p^{I.i}_\alpha u^\alpha_{Ii} - L$. Let $\sigma : U \longrightarrow J^\dagger$ be a local section of $\tau^\dagger$, and $j := \tau^\dagger_0 \circ \sigma : U \longrightarrow J^\infty$. Put $\ldots, \sigma^\alpha_I := \sigma^*(u^\alpha_I) = j^*(u^\alpha_I), \ldots, \sigma^{I.i}_\alpha := \sigma^*(p^{I.i}_\alpha), \ldots$ which are local functions on M. Then, locally,

$$i_{j_1\sigma}\omega_{\mathscr{L}}|_\sigma = [(-\sigma^{I.i}_\alpha{}_{,i} - \delta^I_{Ji}\sigma^{J.i}_\alpha + \partial^I_\alpha L \circ j) d^V u^I_\alpha|_\sigma + (\sigma^\alpha_I{}_{,i} - \sigma^\alpha_{Ii}) d^V p^{I.i}_\alpha|_\sigma] \otimes d^n x,$$

and the PD-Hamilton equations determined by $\omega_{\mathscr{L}}$ read locally

$$\begin{cases} p^{I.i}_\alpha{}_{,i} = \partial^I_\alpha L - \delta^I_{Ji}\, p^{J.i}_\alpha \\ u^\alpha_I{}_{,i} = u^\alpha_{Ii} \end{cases} .$$

We call such equations the *Euler-Lagrange-Hamilton (ELH) equations determined by the Lagrangian density* $\mathscr{L}$. Notice that they are first order PDEs (with an infinite number of dependent variables). Denote by $\mathscr{E}_{ELH} \subset J^\infty\tau^\dagger$ their infinite prolongation. In the following theorem we characterize solutions of $\mathscr{E}_{ELH}$. As a byproduct, we derive the relationship between the ELH equations and the EL equations.

Theorem 4 *A local section $\sigma : U \longrightarrow J^\dagger$ of $\tau^\dagger$ is a solution of the ELH equations determined by the Lagrangian density $\mathscr{L}$ iff it is locally of the form $\sigma = \vartheta \circ j^\infty s$ where 1) $s : U \longrightarrow E$ is a solution of the EL equations $\mathscr{E}_{EL}$ and 2) $\vartheta : U' \longrightarrow J^\dagger$ is a Legendre form for $\mathscr{L}$.*

Proof. Let $\sigma : U \longrightarrow J^\dagger$ be a local section of $\tau^\dagger$. First of all, let σ be of the form $\sigma = \vartheta \circ j$ where 1) $j : U \longrightarrow J^\infty$ is a local section of π_∞ and 2) $\vartheta : U' \longrightarrow J^\dagger$ is a local section of $\tau_0^\dagger : J^\dagger \longrightarrow J^\infty$. Then,

$$\sigma_\alpha^{I.i}{}_{,i} = D_i \vartheta_\alpha^{I.i} \circ j.$$

Therefore, locally,

$$\begin{aligned} i_{j_1\sigma}\omega_{\mathscr{L}}|_\sigma &= [[(-D_i\vartheta_\alpha^{I.i} - \delta_{Ji}^I \vartheta_\alpha^{J.i} + \partial_\alpha^I L) \circ j] d^V u_\alpha^I|_j + (j_I^\alpha{}_{,i} - j_{Ii}^\alpha) d^V p_\alpha^{I.i}|_\sigma] \otimes d^n x \\ &= (\overline{d}\vartheta + d^V \mathscr{L})|_j + (j_I^\alpha{}_{,i} - j_{Ii}^\alpha) d^V p_\alpha^{I.i}|_\sigma \otimes d^n x, \end{aligned}$$

where $\ldots, j_I^\alpha := j^*(u_I^\alpha), \ldots$ and they are local functions on M. Thus, if ϑ is a Legendre form and $j = j_\infty s$ for some local solution $s : U \longrightarrow E$ of the EL equations then, in particular, $j_I^\alpha{}_{,i} = j_{Ii}^\alpha$, $\alpha = 1, \ldots, m$, $i = 1, \ldots, n$, and

$$i_{j_1\sigma}\omega_{\mathscr{L}}|_\sigma = (\overline{d}\vartheta + d^V \mathscr{L})|_j + (j_I^\alpha{}_{,i} - j_{Ii}^\alpha) d^V p_\alpha^{I.i}|_\sigma \otimes d^n x = \boldsymbol{E}(\mathscr{L})|_j = 0.$$

On the other hand, let $\sigma : U \longrightarrow J^\dagger$ be a local section of $\tau^\dagger$ and $j := \tau_0^\dagger \circ \sigma : U \longrightarrow J^\infty$. Locally, there always exists a section $\vartheta : U' \longrightarrow J^\dagger$ of $\tau_0^\dagger$, such that $\sigma = \vartheta \circ j$. Notice, preliminarily, that ϑ is not uniquely determined by σ except for its restriction to $\operatorname{im} j$. If σ is a solution of the ELH equations then, locally,

$$0 = i_{j_1\sigma}\omega_{\mathscr{L}}|_\sigma = (\overline{d}\vartheta + d^V \mathscr{L})|_j + (j_I^\alpha{}_{,i} - j_{Ii}^\alpha) d^V p_\alpha^{I.i}|_\sigma \otimes d^n x.$$

Since $(d^V p_\alpha^{I.i})|_\sigma \otimes d^n x$ and $(\overline{d}\vartheta + d^V \mathscr{L})|_j$ are linearly independent, it follows that

$$\left\{ \begin{array}{l} (\overline{d}\vartheta + d^V \mathscr{L})|_j = 0 \\ j_I^\alpha{}_{,i} = j_{Ii}^\alpha. \end{array} \right. .$$

In particular, $j = j_\infty s$, where $s = \pi_{\infty,0} \circ j$.

Now, let ϑ_0 be a Legendre form for $\mathscr{L}$. Then $d^V \mathscr{L} = \boldsymbol{E}(\mathscr{L}) - \overline{d}\vartheta_0$ and, therefore, $(\overline{d}\vartheta - \overline{d}\vartheta_0 + \boldsymbol{E}(\mathscr{L}))|_j = 0$. Recall that $\overline{d}$ restricts to $j = j_\infty s$ (Remark 2). Thus,

$$\overline{d}|_j (\vartheta - \vartheta_0)|_j = \boldsymbol{E}(\mathscr{L})|_j.$$

In particular, $\boldsymbol{E}(\mathscr{L})|_j$ is $\overline{d}|_j$-exact. In view of Remark 2, this is only possible if $\boldsymbol{E}(\mathscr{L})|_j = 0$, i.e., s is a solution of the EL equations. We conclude that

$$\overline{d}|_j (\vartheta - \vartheta_0)|_j = 0,$$

i.e., $(\vartheta - \vartheta_0)|_j$ is $\overline{d}|_j$-closed. Again in view of Remark 2, this shows that, locally,

$$(\vartheta - \vartheta_0)|_j = \overline{d}|_j \nu|_j = \overline{d}\nu|_j$$

for some $\nu \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-2}$. In particular, we can put $\vartheta = \vartheta_0 + \overline{d}\nu$ and, therefore, ϑ is a Legendre form for $\mathscr{L}$ as well. ■

We now prove a *formal version* of the above theorem. Put $p := \tau^\dagger_{\infty,0} \circ \tau^\dagger_0 : J^\infty\tau^\dagger \longrightarrow J^\infty$.

Theorem 5 *$p(\mathscr{E}_{ELH}) \subset \mathscr{E}_{EL}$ and $p : \mathscr{E}_{ELH} \longrightarrow \mathscr{E}_{EL}$ is a covering of PDEs.*

Proof. In $J^\infty\tau^\dagger$ consider the submanifold $\mathscr{E}_L$ made of ∞th jets of (local) sections $\sigma : U \longrightarrow J^\dagger$ of the form $\sigma = \vartheta \circ j_\infty s$, where $s : U \longrightarrow E$ is a local section of π, and $\vartheta : U' \longrightarrow J^\dagger$ is a local Legendre form. It can be easily checked that $\mathscr{E}_L$ is locally defined by

$$\begin{cases} p_\alpha^{I.i}{}_{|Ki} + \delta^I_{Ji}\, p_\alpha^{J.i}{}_{|K} = D_K(\partial^I_\alpha L) - \delta^I_{\mathsf{O}} D_K \frac{\delta L}{\delta u^\alpha} \\ u^\alpha_{I|K} = u^\alpha_{IK} \end{cases} . \tag{14}$$

Clearly, the Cartan distribution restricts to $\mathscr{E}_L$ and, therefore, $\mathscr{E}_L$ can be interpreted as a PDE. Moreover, it is easily seen from (14) that $\mathscr{E}_L$ covers J^∞ via p. Denote by

$$D'_j = \frac{\partial}{\partial x^j} + u^\alpha_{I|Jj} \frac{\partial}{\partial u^\alpha_{I|J}} + p_\alpha^{I.i}{}_{|Jj} \frac{\partial}{\partial p_\alpha^{I.i}{}_{|J}}$$

the jth total derivative on $J^\infty\tau^\dagger$, $j = 1, \ldots, n$. $\mathscr{E}_{ELH}$ is locally defined by

$$\begin{cases} p_\alpha^{I.i}{}_{|Ki} = D'_K(\partial^I_\alpha L) - \delta^I_{Ji}\, p_\alpha^{J.i}{}_{|K} \\ u^\alpha_{I|Ki} = u^\alpha_{Ii|K} \end{cases} , \tag{15}$$

which is equivalent to

$$\begin{cases} p_\alpha^{I.i}{}_{|Ki} = D_K(\partial^I_\alpha L) - \delta^I_{Ji}\, p_\alpha^{J.i}{}_{|K} \\ u^\alpha_{I|K} = u^\alpha_{IK} \end{cases} .$$

Moreover, on $\mathscr{E}_{ELH}$

$$(-)^{|I|} p_\alpha^{I.i}{}_{|KIi} = D_K \frac{\delta L}{\delta u^\alpha} - (-)^{|I|} \delta^I_{Ji}\, p_\alpha^{J.i}{}_{|KI} = D_K \frac{\delta L}{\delta u^\alpha} + (-)^{|I|} p_\alpha^{I.i}{}_{|KIi},$$

and, therefore, $D_K \frac{\delta L}{\delta u^\alpha} = 0$, $\alpha = 1, \ldots, m$. It then follows from (14), that $\mathscr{E}_{ELH} = \mathscr{E}_L \cap p^{-1}(\mathscr{E}_{EL})$. In particular, $p : \mathscr{E}_{ELH} \longrightarrow \mathscr{E}_{EL}$ is the "restriction" of $p : \mathscr{E}_L \longrightarrow J^\infty$ to $\mathscr{E}_{EL} \subset J^\infty$ and, therefore, is a covering. ■

6 Natural Transformations of the Euler-Lagrange-Hamilton Equations

Properties of Legendre forms discussed in Remark 1 correspond to specific properties of the ELH equations which we discuss in this section.

First of all, notice that the ELH equations are canonically associated to a Lagrangian density. But, how do the ELH equations change when changing the Lagrangian density into a $\overline{d}$-cohomology class? In particular, does an action functional uniquely determine a system of ELH equations or not? In order to answer these questions consider $\vartheta \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$. ϑ determines an automorphism $\Psi_\vartheta : J^\dagger \longrightarrow J^\dagger$ of the fiber bundle $\tau_0^\dagger$ via

$$\Psi_\vartheta(P) := P - \vartheta_\theta, \quad P \in J^\dagger,\ \theta = \tau_0^\dagger(P) \in J^\infty.$$

In particular, $\tau_0^\dagger \circ \Psi_\vartheta = \tau_0^\dagger$. Clearly, $\Psi_\vartheta^{-1} = \Psi_{-\vartheta}$.

Lemma 6 $\Psi_\vartheta^*(\omega_{\mathscr{L}}) = \omega_{\mathscr{L}} - \tau_0^{\dagger *}(d\vartheta)$.

Proof. Compute,

$$\begin{aligned}
\Psi_\vartheta^*(\omega_{\mathscr{L}}) &= \Psi_\vartheta^*(d\Theta_{\mathscr{L}}) \\
&= d\Psi_\vartheta^*(\Theta_{\mathscr{L}}) \\
&= d[(\Psi_\vartheta^* \circ \Sigma^*)(\Theta) + (\Psi_\vartheta^* \circ \tau_0^{\dagger *})(\mathscr{L})] \\
&= d[(\Psi_\vartheta^* \circ \Sigma^*)(\Theta) + (\tau_0^\dagger \circ \Psi_\vartheta)^*(\mathscr{L})] \\
&= d[(\Psi_\vartheta^* \circ \Sigma^*)(\Theta) + \tau_0^{\dagger *}(\mathscr{L})].
\end{aligned}$$

Now, since, locally, $\ldots, \Psi_\vartheta^*(p_\alpha^{I.i}) = p_\alpha^{I.i} - \vartheta_\alpha^{I.i}, \ldots$, we have

$$\begin{aligned}
(\Psi_\vartheta^* \circ \Sigma^*)(p_\alpha^{I.i}) &= p_\alpha^{I.i} - \vartheta_\alpha^{I.i}, \\
(\Psi_\vartheta^* \circ \Sigma^*)(p) &= (p_\alpha^{I.i} - \vartheta_\alpha^{I.i}) u_{Ii}^\alpha.
\end{aligned}$$

Thus, locally

$$\begin{aligned}
(\Psi_\vartheta^* \circ \Sigma^*)(\Theta) &= (p_\alpha^{I.i} - \vartheta_\alpha^{I.i}) du_I^\alpha d^{n-1}x_i - (p_\alpha^{I.i} - \vartheta_\alpha^{I.i}) u_{Ii}^\alpha d^n x \\
&= \Sigma^*(\Theta) - \tau_0^{\dagger *}(\vartheta).
\end{aligned}$$

We conclude that

$$\begin{aligned}
\Psi_\vartheta^*(\omega_{\mathscr{L}}) &= d[(\Psi_\vartheta^* \circ \Sigma^*)(\Theta) + \tau_0^{\dagger *}(\mathscr{L})] \\
&= d[\Sigma^*(\Theta) - \tau_0^{\dagger *}(\vartheta) + \tau_0^{\dagger *}(\mathscr{L})] \\
&= \omega_{\mathscr{L}} - \tau_0^{\dagger *}(d\vartheta).
\end{aligned}$$

■

Theorem 7 *Let $\mathscr{L}' = \mathscr{L} + \overline{d}\varrho$, $\varrho \in \overline{\Lambda}^{n-1}$, be another Lagrangian density (thus, $\mathscr{L}'$ determines the same EL equations as $\mathscr{L}$). Then $\Psi^*_{d^V\varrho}(\omega_{\mathscr{L}}) = \omega_{\mathscr{L}'}$.*

Proof. Notice, preliminarily, that

$$\begin{aligned}
\tau_0^{\dagger *}(dd^V\varrho) &= \tau_0^{\dagger *}(\overline{d}d^V\varrho) \\
&= -\tau_0^{\dagger *}(d^V\overline{d}\varrho) \\
&= -\tau_0^{\dagger *}(d\overline{d}\varrho) \\
&= -d\tau_0^{\dagger *}(\overline{d}\varrho).
\end{aligned}$$

Therefore, in view of the above lemma,

$$\begin{aligned}
\Psi^*_{d^V\varrho}(\omega_{\mathscr{L}}) &= \omega_{\mathscr{L}} - \tau_0^{\dagger *}(dd^V\varrho) \\
&= d[\Sigma^*(\Theta) + \tau_0^{\dagger *}(\mathscr{L})] + d\tau_0^{\dagger *}(\overline{d}\varrho) \\
&= d[\Sigma^*(\Theta) + \tau_0^{\dagger *}(\mathscr{L} + \overline{d}\varrho)] \\
&= d\Theta_{\mathscr{L}'} \\
&= \omega_{\mathscr{L}'}.
\end{aligned}$$

■

Corollary 8 *An action $[\mathscr{L}] \in \overline{H}^n$, $\mathscr{L} \in \overline{\Lambda}^n$, uniquely determines a system of ELH equations, modulo isomorphisms of PD-Hamiltonian systems.*

We conclude that the ELH equations are basically determined by the sole action functional and not a specific Lagrangian density.

Theorem 9 *Let $\vartheta \in \mathscr{C}\Lambda^1 \otimes \overline{\Lambda}^{n-1}$ be $\overline{d}$-closed, hence $\overline{d}$-exact. Then, for every Lagrangian density $\mathscr{L} \in \overline{\Lambda}^n$, Ψ_ϑ is a symmetry of the ELH equations determined by $\mathscr{L}$ in the sense that $j_\infty\Psi_\vartheta : J^\infty\tau^\dagger \longrightarrow J^\infty\tau^\dagger$ preserves $\mathscr{E}_{ELH}$.*

Proof. By definition of infinite prolongations of a PDE and infinite prolongation of a morphism of bundles, it is enough to prove that $j_1\Psi_\vartheta : J^1\tau^\dagger \longrightarrow J^1\tau^\dagger$ preserves $\mathscr{E}^{(0)}_{ELH} := \mathscr{E}^{(0)}_{\omega_{\mathscr{L}}} \subset J^1\tau^\dagger$. Notice, preliminarily, that, in view of the proof of Theorem 4, we have

$$(j_1\tau_0^\dagger)(\mathscr{E}^{(0)}_{ELH}) \subset \operatorname{im}\mathscr{C} \subset J^1\pi_\infty.$$

Now, let $c \in \mathscr{E}^{(0)}_{ELH}$, $P := \tau^\dagger_{1,0}(c)$ and $\xi \in T_P J^\dagger$ be a tangent vector, vertical with respect to $\tau^\dagger$. Consider also $c' := (j_1\Psi_\vartheta)(c)$, $P' := \Psi_\vartheta(P) = \tau^\dagger_{1,0}(c')$ and $\xi' := d\Psi_\vartheta(\xi)$. In

particular, $\xi' \in T_{P'}J^\dagger$ is vertical with respect to $\tau^\dagger$ as well. Let us prove that $c' \in \mathscr{E}^{(0)}_{ELH}$. In view of Lemma 6,

$$\Psi^*_\vartheta(\omega_{\mathscr{L}}) = \omega_{\mathscr{L}} - \tau_0^{\dagger *}(d\vartheta) = \omega_{\mathscr{L}} - \tau_0^{\dagger *}(d^V\vartheta).$$

Compute

$$i_{\xi'}i_{c'}(\omega_{\mathscr{L}})_{P'} = i_\xi i_c \Psi^*_\vartheta(\omega_{\mathscr{L}})_P = i_\xi i_c(\omega_{\mathscr{L}})_P - i_\xi i_c[\tau_0^{\dagger *}(d^V\vartheta)]_P = -i_{\xi''}i_{\mathscr{C}_\theta}(d^V\vartheta)_\theta = 0,$$

where $\theta = \tau_0^\dagger(P) \in J^\infty$ and $\xi'' = (d\tau_0^\dagger)(\xi) \in T_\theta J^\infty$ is a tangent vector, vertical with respect to π_∞. It follows from the arbitrariness of ξ', that $i_{c'}(\omega_{\mathscr{L}})_{P'} = 0$. ■

7 Hamiltonian Formalism

In this section we present our proposal of an Hamiltonian formalism for higher order Lagrangian field theories. Such proposal is free from ambiguities in that it depends only on the choice of a Lagrangian density and its order. Moreover, $\overline{d}$-cohomologous Lagrangians of the same order determine equivalent "Hamiltonian theories".

First of all, we define a "finite dimensional version" of the ELH equations (see also [17]). In order to do this, notice that, in view of Remark 3, for all $k \geq 0$, W_k is canonically isomorphic to the $C^\infty(J^{k+1})$-module of sections of the induced bundle $\pi^\circ_{k+1,k}(J^\dagger\pi_k) \longrightarrow J^{k+1}$. We conclude that $J^\dagger_k \longrightarrow J^\infty$ is canonically isomorphic to the pull-back bundle $\pi^\circ_{\infty,k}(J^\dagger\pi_k) \longrightarrow J^\infty$, $k \geq 0$. Notice that the coordinates $\ldots, p^{I.i}_\alpha, \ldots$, $|I| \leq k$, on $J^\dagger_k$ identify with the pull-backs of the corresponding natural coordinates on $J^\dagger\pi_k$ which we again denote by $\ldots, p^{I.i}_\alpha, \ldots$.

Now, let $\mathscr{L} \in \overline{\Lambda}^n$ be a Lagrangian density of order $l+1$, i.e., $\mathscr{L} \in \overline{\Lambda}^n \cap \Lambda(J^{l+1})$. Let ω'_l be the pull-back of $\omega_{\mathscr{L}}$ onto $J^\dagger_l$. ω'_l is a PD-Hamiltonian system on $J^\dagger_l \longrightarrow M$, and it is locally given by

$$\omega'_l = \sum_{|I|\leq l} dp^{I.i}_\alpha du^\alpha_I d^{n-1}x_i - dE_l d^n x,$$

where $E_l = \sum_{|I|\leq l} p^{I.i}_\alpha u^\alpha_{Ii} - L$ is the restriction of $E_{\mathscr{L}}$ to $J^\dagger_l$. Notice that ω'_l is also the pull-back via $J^\dagger_l \longrightarrow \pi^\circ_{l+1,l}(J^\dagger\pi_l)$ of a (unique) PD-Hamiltonian system ω_l on $\pi^\circ_{l+1,l}(J^\dagger\pi_l) \longrightarrow M$. ω_l is locally given by the same formula as ω'_l and it is a constrained PD-Hamiltonian system, i.e., its first constraint bundle $\mathscr{P} \longrightarrow M$ is a proper subbundle of $\pi^\circ_{l+1,l}(J^\dagger\pi_l) \longrightarrow M$. Let us compute it. Let $P \in \pi^\circ_{l+1,l}(J^\dagger\pi_l)$ and $\theta := \pi^\circ_{l+1,l}(\tau_0^\dagger\pi_l)(P) \in J^{l+1}$. Then $P \in \mathscr{P}$ iff there exists c in the first jet bundle of $\pi^\circ_{l+1,l}(J^\dagger\pi_l) \longrightarrow M$ such that $i_c(\omega_l)_P = 0$, i.e., iff there exist real numbers $\ldots, c^a_{I\cdot i}, \ldots, c^{I.i}_{\alpha}{}_{\cdot j}, \ldots$, $|I| \leq l$, such that

$$\begin{cases} c^{I.i}_\alpha{}_{\cdot i} = (\partial^I_\alpha L)(\theta) - \delta^I_{Ji}\, P^{J.i}_\alpha, & |I| \leq l+1 \\ c^\alpha_{J\cdot i} = P^\alpha_{Ji}, & |J| \leq l \end{cases}$$

where we put $c_\alpha^{I.i}{}_{\cdot i} = 0$ for $|I| = l+1$, and $\ldots, P^\alpha_{Ji} := u^\alpha_{Ji}(P), \ldots, P^{K.i}_\alpha := p^{K.i}_\alpha(P), \ldots$, $|J|, |K| \leq l$, $\alpha = 1, \ldots, m$. Thus, for $|I| = l+1$, P should be a solution of the system

$$\partial^I_\alpha L - \delta^I_{Ji} p^{J.i}_\alpha = 0, \quad |I| = l+1. \tag{16}$$

Equations (16) define $\mathscr{P}$ locally.

Remark 4 *$\mathscr{P}$ is a submanifold of $\pi^\circ_{l+1,l}(J^\dagger\pi_l)$ of the same dimension as $J^\dagger\pi_l$, and $\mathscr{P} \longrightarrow J^{l+1}$ an affine subbundle of $\pi^\circ_{l+1,l}(J^\dagger\pi_l) \longrightarrow J^{l+1}$.*

Let $\mathscr{P}_0$ be the image of $\mathscr{P}$ under the projection $\pi^\circ_{l+1,l}(J^\dagger\pi_l) \longrightarrow J^\dagger\pi_l$.

Assumption 1 *We assume $\mathscr{P}_0$ to be a submanifold of $J^\dagger\pi_l$ and $\tau^\dagger\pi_l|_{\mathscr{P}_0} : \mathscr{P}_0 \longrightarrow M$ to be a smooth subbundle of $\tau^\dagger\pi_l$. We also assume that the projection $q : \mathscr{P} \longrightarrow \mathscr{P}_0$ is a smooth submersion with connected fibers.*

Notice that, as usual, all the above regularity conditions are true if we restrict all the involved maps to suitable open subsets.

The following commutative diagram summarizes the above described picture:

$$\begin{tikzcd}
J^\dagger \arrow[d] & J^\dagger_l \arrow[l, hook'] \arrow[d] \arrow[dl] & \\
J^\infty \arrow[d] & \pi^\circ_{l+1,l}(J^\dagger\pi_l) \arrow[d] \arrow[dl] & \mathscr{P} \arrow[l, hook'] \arrow[d, "q"] \\
J^{l+1} \arrow[d] & J^\dagger\pi_l \arrow[dl, "\tau_0^\dagger\pi_l"'] \arrow[ddl, "\tau^\dagger\pi_l"] & \mathscr{P}_0 \arrow[l, hook'] \arrow[ddll] \\
J^l \arrow[d, "\pi_l"'] & & \\
M & &
\end{tikzcd} .$$

Theorem 10 *Under the regularity Assumption 1, there exists a unique PD-Hamiltonian system ω_0 on $\mathscr{P}_0 \longrightarrow M$, such that $i^*_{\mathscr{P}}(\omega_l)$ is the pull-back of ω_0 via $q : \mathscr{P} \longrightarrow \mathscr{P}_0$.*

Proof. Since $q : \mathscr{P} \longrightarrow \mathscr{P}_0$ has connected fibers and $i^*_{\mathscr{P}}(\omega)$ is a closed form, it is enough to prove that $i_{\overline{Y}} i^*_{\mathscr{P}}(\omega_l) = 0$ for all vector fields $\overline{Y} \in \mathrm{D}(\mathscr{P})$ vertical with respect to q. Let $Y \in \mathrm{D}(\pi^{\circ}_{l+1,l}(J^{\dagger}\pi_l))$ be vertical with respect to $\pi^{\circ}_{l+1,l}(J^{\dagger}\pi_l) \longrightarrow J^{\dagger}\pi_l$, and $\overline{Y} := Y|_{\mathscr{P}}$. Then $\overline{Y}$ is locally of the form

$$\overline{Y} = \sum_{|K|=l+1} Y^{\beta}_{K} \partial^{K}_{\beta}|_{\mathscr{P}},$$

for some $\ldots, Y^{\beta}_{K}, \ldots$ local functions on $\mathscr{P}$. Now $\overline{Y} \in \mathrm{D}(\mathscr{P})$ iff, locally,

$$\sum_{|I|=l+1} Y^{\beta}_{K} \partial^{K}_{\beta} \partial^{I}_{\alpha} L|_{\mathscr{P}} = 0.$$

Compute

$$\overline{Y}(E_l|_{\mathscr{P}}) = \sum_{|K|=l+1} Y^{\beta}_{K} \partial^{K}_{\beta} E_l|_{\mathscr{P}} = \sum_{|I|=l+1} Y^{\alpha}_{I} (\delta^{I}_{Ji} p^{J.i}_{\alpha} - \partial^{I}_{\alpha} L)|_{\mathscr{P}} = 0.$$

Thus $E_l|_{\mathscr{P}}$ is the pull-back via q of a (unique) local function H on $\mathscr{P}_0$. Moreover,

$$i_{\overline{Y}} i^*_{\mathscr{P}}(\omega_l) = -\overline{Y}(E_l|_{\mathscr{P}}) d^n x = 0.$$

It follows from the arbitrariness of $\overline{Y}$ that $i^*_{\mathscr{P}}(\omega_l)$ is the pull-back via q of the PD-Hamiltonian system ω_0 on $\mathscr{P}_0 \longrightarrow M$ locally defined as

$$\omega_0 = \sum_{|I|\leq l} i^*_{\mathscr{P}_0}(dp^{I.i}_{\alpha} du^{\alpha}_{I}) d^{n-1}x_i - dH d^n x.$$

■

Definition 2 ω_0 *is called the* PD-Hamiltonian system determined by the $(l+1)$th order Lagrangian density $\mathscr{L}$*, and the corresponding PD-Hamilton equations are the* Hamilton-de Donder-Weyl (HDW) equations determined by $\mathscr{L}$*.*

Definition 3 *A Lagrangian density $\mathscr{L}$ of order $l+1$ is* regular at the order $l+1$ *iff the map $\mathscr{P} \longrightarrow J^{\dagger}\pi_l$ has maximum rank.*

The Lagrangian density $\mathscr{L}$ of order $l+1$ is regular at the order $l+1$ iff the matrix

$$\mathbf{H}(L)(\theta) := \left\| (\partial^{K}_{\beta} \partial^{I}_{\alpha} L)(\theta) \right\|^{(\alpha,I)}_{(\beta,K)}, \quad |I|, |K| = l+1,$$

where the pairs (α, I) and (β, K) are understood as single indexes, has maximum rank at every point $\theta \in J^{l+1}$. In its turn, this implies that $\mathscr{P}_0$ is an open submanifold of $J^{\dagger}\pi_l$

and, in view of Remark 4 and Assumption 1, $q : \mathscr{P} \longrightarrow \mathscr{P}_0$ is a diffeomorphism. In particular, ω_0 is a PD-Hamiltonian system on an open subbundle of $\tau^\dagger \pi_l$ locally given by

$$\omega_0 = \sum_{|I| \leq l} dp_\alpha^{I.i} du_I^\alpha d^{n-1} x_i - dH d^n x,$$

where, now, H is a local function on $J^\dagger \pi_l$. In this case, as expected, the HDW equations read locally

$$\begin{cases} p_\alpha^{I.i}{}_{,i} = -\frac{\partial H}{\partial u_I^\alpha} \\ u_I^\alpha{}_{,i} = \frac{\partial H}{\partial p_\alpha^{I.i}} \end{cases}.$$

Notice that the HDW equations are canonically associated to a Lagrangian density and its order and no additional structure is required to define them. Moreover, in view of Theorem 7, two Lagrangian densities of the same order determining the same system of EL equations, also determine equivalent HDW equations. Finally, to write down the HDW equations there is no need of a distinguished Legendre transform. Actually, the emergence of ambiguities in all Hamiltonian formalisms for higher order field theories proposed in the literature seems to rely on the common attempt to define first a higher order analogue of the Legendre transform and, only thereafter, the "Hamiltonian theory". In the next section we present our own point of view on the Legendre transform in higher order Lagrangian field theories.

8 The Legendre Transform

Keeping the same notations as in the previous section, denote by ${}^l\mathscr{E}_{ELH}$ the infinite prolongation of the PD-Hamilton equations determined by ω_l and by $p' : \pi_{l+1,l}^\circ(J^\dagger \pi_l) \longrightarrow E$ the natural projection.

Proposition 11 *$(j_\infty p')({}^l\mathscr{E}_{ELH}) \subset \mathscr{E}_{EL}$ and $j_\infty p' : {}^l\mathscr{E}_{ELH} \longrightarrow \mathscr{E}_{EL}$ is a covering.*

Proof. The proof is the finite dimensional version of the proof of Theorem 5 and will be omitted (see also [17]). ■

Denote also by $\mathscr{E}_H^{\mathscr{P}}$ the infinite prolongation of the PD-Hamilton equations determined by $i_{\mathscr{P}}^*(\omega_l)$ and by $\mathscr{E}_H$ the infinite prolongation of the HDW equations.

Proposition 12 *$(j_\infty q)(\mathscr{E}_H^{\mathscr{P}}) \subset \mathscr{E}_H$ and $j_\infty q : \mathscr{E}_H^{\mathscr{P}} \longrightarrow \mathscr{E}_H$ is a covering.*

Proof. It immediately follows from Theorem 10 and Proposition 2. ■

Notice that, in view of Propositions 3, 11 and 12, there is a diagram of morphisms of PDEs,

$$\begin{tikzcd} {}^{l}\mathscr{E}_{ELH} \arrow[r, hook] \arrow[d, "j_\infty p'"'] & \mathscr{E}_H^{\mathscr{P}} \arrow[d, "j_\infty q"] \\ \mathscr{E}_{EL} & \mathscr{E}_H \end{tikzcd} , \tag{17}$$

whose vertical arrows are coverings. Therefore, the inclusion ${}^{l}\mathscr{E}_{ELH} \subset \mathscr{E}_H^{\mathscr{P}}$ may be understood as a non local morphism of $\mathscr{E}_{EL}$ into $\mathscr{E}_H$. We interpret such morphism as Legendre transform according to the following

Definition 4 *We call diagram (17) the* Legendre transform determined by the Lagrangian density *$\mathscr{L}$.*

Any Legendre form of order l, $\vartheta : J^\infty \longrightarrow J_l^\dagger \longrightarrow \pi^\circ_{l+1,l}(J^\dagger\pi_l)$, determines a section $j_\infty\vartheta|_{\mathscr{E}_{EL}} : \mathscr{E}_{EL} \longrightarrow {}^{l}\mathscr{E}_{ELH}$ of the covering $j_\infty p' : {}^{l}\mathscr{E}_{ELH} \longrightarrow \mathscr{E}_{EL}$ and, therefore, via composition with $j_\infty q$, a concrete map $\mathscr{E}_{EL} \longrightarrow \mathscr{E}_H$. Nevertheless, among these maps, there is no distinguished one.

We now prove that, if $\mathscr{L}$ is regular at the order $l+1$, then $\mathscr{E}_H$ itself covers $\mathscr{E}_{EL}$. This result should be interpreted as the higher order analogue of the theorem stating the equivalence of EL equations and HDW equations for first order theories with regular Lagrangian (see, for instance, [11]). Let us first prove the following

Lemma 13 *If $\mathscr{L}$ is regular at the order $l+1$, then ${}^{l}\mathscr{E}_{ELH} = \mathscr{E}_H^{\mathscr{P}}$.*

Proof. The proof is in local coordinates. Let $\sigma : U \longrightarrow \pi^\circ_{l+1,l}(J^\dagger\pi_l)$ be a local section of $\pi^\circ_{l+1,l}(J^\dagger\pi_l) \longrightarrow M$. Suppose $\operatorname{im}\sigma \subset \mathscr{P}$. Then, locally,

$$\partial^I_\alpha L \circ \sigma - \delta^I_{Jj}\sigma^{J.j}_\alpha = 0, \quad |I| = l+1.$$

Now, $i_{j_1\sigma}\omega_l|_\sigma$ is locally given by

$$i_{j_1\sigma}\omega_l|_\sigma = \Big[\textstyle\sum_{|I|\leq l+1}(-\sigma^{I.i}_\alpha{}_{,i} - \delta^I_{Jj}\sigma^{J.j}_\alpha + \partial^I_\alpha L\circ\sigma)d^V u^\alpha_I + \sum_{|I|\leq l}(\sigma^\alpha_I{}_{,i} - \sigma^\alpha_{Ii})d^V p^{I.i}_\alpha\Big]|_\sigma \otimes d^n x.$$

As already outlined, the annihilator of $\mathrm{D}(\mathscr{P})$ in $\Lambda^1(\pi^\circ_{l+1,l}(J^\dagger\pi_l))|_{\mathscr{P}}$ is locally spanned by 1-forms

$$\lambda^I_\alpha := d(\partial^I_\alpha L - \delta^I_{Jj}p^{J.j}_\alpha)|_{\mathscr{P}}, \quad |I| = l+1.$$

Therefore, $i_{j_1\sigma}i^*_{\mathscr{P}}(\omega_l)|_\sigma = 0$ iff, locally,

$$i_{j_1\sigma}\omega_l|_\sigma = \sum_{|I|=l+1} f^a_I \underline{\lambda}^I_\alpha|_\sigma \otimes d^n x, \tag{18}$$

for some local functions $\ldots, f_I^\alpha, \ldots$ on $\operatorname{im}\sigma$, where

$$\underline{\lambda}_\alpha^I := d^V(\partial_\alpha^I L - \delta_{Jj}^I p_\alpha^{J.j})|_{\mathscr{P}} = \textstyle\sum_{|K|\leq l+1}(\partial_\beta^K \partial_\alpha^I L\, d^V u_K^\beta - \delta_{Jj}^I d^V p_\alpha^{Jj})|_{\mathscr{P}}, \quad |I| = l+1.$$

Equations (18) read

$$\begin{array}{c} \sum_{|I|\leq l+1}(-\sigma_\alpha^{I.i},_i + \partial_\alpha^I L \circ \sigma - \delta_{Jj}^I \sigma_\alpha^{J.j} - \sum_{|K|=l+1} f_K^\beta \partial_\beta^K \partial_\alpha^I L \circ \sigma) d^V u_I^\alpha|_\sigma \\ + \sum_{|I|<l}(\sigma_{I,i}^\alpha - \sigma_{Ii}^\alpha) d^V p_\alpha^{I.i}|_\sigma + \sum_{|I|=l}(\sigma_I^\alpha,_i - \sigma_{Ii}^\alpha + \frac{I[i]+1}{l+1} f_{Ii}^\alpha) d^V p_\alpha^{I.i}|_\sigma = 0 \end{array},$$

where $I[i]$ is the number of times the index i appears in the multiindex I. Since the vertical forms $\ldots, d^V u_I^\alpha|_\sigma, \ldots, d^V p_\alpha^{I.i}|_\sigma, \ldots$ are linearly independent, $i_{j_1\sigma} i_{\mathscr{P}}^*(\omega_l)|_\sigma = 0$ iff, locally,

$$\left\{ \begin{array}{ll} -\sigma_\alpha^{I.i},_i + \partial_\alpha^I L \circ \sigma - \delta_{Jj}^I \sigma_\alpha^{J.j} - \sum_{|K|=l+1} f_K^\beta \partial_\beta^K \partial_\alpha^I L \circ \sigma = 0, & |I| \leq l+1 \\ \sigma_{I,i}^\alpha - \sigma_{Ii}^\alpha = 0, & |I| < l \\ \sigma_I^\alpha,_i - \sigma_{Ii}^\alpha + \frac{I[i]+1}{l+1} f_{Ii}^\alpha = 0, & |I| = l \end{array} \right., \tag{19}$$

for some $\ldots, f_I^\alpha, \ldots$. It follows from the third of Equations (19) that

$$f_{Ii}^\alpha = -\tfrac{l+1}{I[i]+1}(\sigma_I^\alpha,_i - \sigma_{Ii}^\alpha), \quad |I| = l. \tag{20}$$

Moreover, since $\operatorname{im}\sigma \subset \mathscr{P}$, the first equation, for $|I| = l+1$, gives

$$0 = \sum_{|K|=l+1} f_K^\beta \partial_\beta^K \partial_\alpha^I L \circ \sigma = \sum_{|J|=l} \tfrac{J[j]+1}{l+1} f_{Jj}^\beta \partial_\beta^{Jj} \partial_\alpha^I L \circ \sigma = -\sum_{|J|=l} (\sigma_J^\beta,_j - \sigma_{Jj}^\beta) \partial_\beta^{Jj} \partial_\alpha^I L \circ \sigma,$$

and, in view of the regularity of $\mathscr{L}$ and Equations (20),

$$\sigma_I^\alpha,_i - \sigma_{Ii}^\alpha = f_{Ii}^\alpha = 0, \quad |I| = l.$$

Substituting again into (19), we finally find that the PD-Hamilton equations $i_{j_1\sigma} i_{\mathscr{P}}^*(\omega_l)|_\sigma = 0$ are locally equivalent to equations

$$\left\{ \begin{array}{ll} p_\alpha^{I.i},_i = \partial_\alpha^I L - \delta_{Jj}^I p_\sigma^{J.j}, & |I| \leq l+1 \\ u_I^\alpha,_i = u_{Ii}^\alpha, & |I| \leq l \end{array} \right.,$$

which are the PD-Hamilton equations determined by ω_l. ■

Now, suppose that $\mathscr{L}$ is regular at the order $l+1$. Then, as already mentioned in the previous section, $q : \mathscr{P} \longrightarrow \mathscr{P}_0$ is a diffeomorphism, and $q^*(\omega_0) = i_{\mathscr{P}}^*(\omega_l)$. Therefore, $j_\infty q : \mathscr{E}_H^{\mathscr{P}} \longrightarrow \mathscr{E}_H$ is an isomorphism of PDEs and the Legendre transform (17) reduces to

$$\begin{tikzcd} {}^l\mathscr{E}_{ELH} \arrow[r, equal] \arrow[d, "j_\infty p'"'] & \mathscr{E}_H^{\mathscr{P}} \arrow[d, "j_\infty q", "\wr"'] \\ \mathscr{E}_{EL} & \mathscr{E}_H \end{tikzcd}.$$

Moreover, $J^{\dagger}\pi_l$ maps to E via $\pi_{l,0} \circ \tau_0^{\dagger}\pi_l$ and such map is a morphism of bundles (over M). The induced morphism $J^{\infty}\tau_0^{\dagger}\pi_l \longrightarrow J^{\infty}$ restricts to a morphism of PDEs, $\kappa : \mathscr{E}_H \longrightarrow J^{\infty}$, locally defined as $\kappa^*(u_K^{\alpha}) = u_{\mathsf{O}|K}^{\alpha}$, $|K| \geq 0$. It is easy to show that diagram

$$\begin{tikzcd}
{}^{l}\mathscr{E}_{ELH} \arrow[rr, equal] \arrow[d, "j_\infty p'"'] & & \mathscr{E}_H^{\mathscr{P}} \arrow[d, "\wr"', "j_\infty q"] \\
\mathscr{E}_{EL} \arrow[r, hook] & J^\infty & \mathscr{E}_H \arrow[l, "\kappa"']
\end{tikzcd} ,$$

commutes, so that $\kappa = j_\infty p' \circ (j_\infty q|_{\mathscr{E}_H})^{-1}$. Consequently, $\kappa(\mathscr{E}_H) \subset \mathscr{E}_{EL}$ and $\kappa : \mathscr{E}_H \longrightarrow \mathscr{E}_{EL}$ is a covering. Summarizing, we have have proved the following

Theorem 14 *If $\mathscr{L}$ is regular at the order $l+1$, then $\mathscr{E}_H$ covers $\mathscr{E}_{EL}$.*

Finally, it should be mentioned that in most cases, even if the Lagrangian density is not regular, $\mathscr{E}_H$ covers $\mathscr{E}_{EL}$ via κ and, therefore, $\mathscr{E}_H^{\mathscr{P}}$ itself covers $\mathscr{E}_{EL}$ (see the example in the next section).

9 An Example: The Korteweg-de Vries Action

The celebrated Korteweg-de Vries (KdV) equation

$$\phi_t - 6\phi\phi_x + \phi_{xxx} = 0 \tag{21}$$

can be derived from a variational principle as follows. Introduce the "potential" u by putting $u_x = \phi$. Equation (21) becomes the fourth order non-linear equation

$$u_{tx} - 6u_x u_{xx} + u_{xxxx} = 0 \tag{22}$$

for sections of the trivial bundle $\pi : \mathbb{R}^2 \times \mathbb{R} \ni (t,x;u) \longmapsto (t,x) \in \mathbb{R}^2$. In its turn, (22) is the EL equation determined by the action functional

$$\int (u_x^3 - \tfrac{1}{2}u_x u_t + \tfrac{1}{2}u_{xx}^2) dt dx.$$

Choose the second order Lagrangian density

$$\mathscr{L} = (u_x^3 - \tfrac{1}{2}u_x u_t + \tfrac{1}{2}u_{xx}^2) dt dx. \tag{23}$$

Since the matrix

$$\mathbf{H}(L) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

has rank 1, $\mathscr{L}$ is not regular. Let $t, x, u, u_t, u_x, p^{\cdot t}, p^{\cdot x}, \dots p^{i.j}, \dots$ be natural coordinates on $J^\dagger\pi_1$, $i, j = t, x$. Then

$$\omega_1 = dp^{\cdot t}dudx - dp^{\cdot x}dudt + dp^{t.t}du_tdx - dp^{t.x}du_tdt + dp^{x.t}du_xdx - dp^{x.x}du_xdt - dE_{\mathrm{KdV}}dtdx$$

where

$$E_{\mathrm{KdV}} := p^{\cdot t}u_t + p^{\cdot x}u_x + p^{t.t}u_{tt} + (p^{t.x} + p^{x.t})u_{tx} + p^{x.x}u_{xx} - u_x^3 + \tfrac{1}{2}u_xu_t - \tfrac{1}{2}u_{xx}^2.$$

Accordingly, ${}^1\mathscr{E}_{ELH}$ reads

$$ {}^1\mathscr{E}_{ELH} : \begin{cases} p^{\cdot t}{}_{,t} + p^{\cdot x}{}_{,x} = 0 & \\ p^{t.t}{}_{,t} + p^{t.x}{}_{,x} = -\frac{1}{2}u_x & \\ p^{x.t}{}_{,t} + p^{x.x}{}_{,x} = 3u_x^2 - \frac{1}{2}u_t & \\ u_{,i} = u_i & i = t, x \\ u_{i,j} = u_{ij} & i, j = t, x \\ p^{t.t} = 0 & \\ p^{t.x} + p^{x.t} = 0 & \\ p^{x.x} - u_{xx} = 0 & \end{cases} . \tag{24}$$

which clearly cover (22). Notice that the last three equations in (24) define $\mathscr{P}$. Thus, $\mathscr{P}$ is coordinatized by $t, x, u, u_t, u_x, u_{tt}, u_{tx}, p^{t.x}, p^{x.x}$ and

$$i^*_{\mathscr{P}}(\omega_1) = dp^{\cdot t}dudx - dp^{\cdot x}dudt - dp^{t.x}(du_tdt + du_xdx) - dp^{x.x}du_xdt - dE_{\mathrm{KdV}}|_{\mathscr{P}}dtdx,$$

where

$$E_{\mathrm{KdV}}|_{\mathscr{P}} := p^{\cdot t}u_t + p^{\cdot x}u_x + \tfrac{1}{2}(p^{x.x})^2 - u_x^3 + \tfrac{1}{2}u_xu_t.$$

Accordingly, $\mathscr{E}_H^{\mathscr{P}}$ reads

$$\mathscr{E}_H^{\mathscr{P}} : \begin{cases} p^{\cdot t}{}_{,t} + p^{\cdot x}{}_{,x} = 0 & \\ p^{t.x}{}_{,x} = -\frac{1}{2}u_x & \\ p^{t.x}{}_{,t} + p^{x.x}{}_{,x} = -3u_x^2 + \frac{1}{2}u_t & \\ u_{,i} = u_i & i = t, x \\ u_{t,x} = u_{x,t} & \\ u_{x,x} = p^{x.x} & \end{cases} .$$

Notice that, even if the Lagrangian density is not regular, and variables u_{tt}, u_{tx} are undetermined, $\mathscr{E}_H^{\mathscr{P}}$ covers (22). Finally, $\mathscr{P}$ is defined by the sixth and the seventh equations in (24) and, therefore, it is coordinatized by $t, x, u, u_t, u_x, p^{t.x}, p^{x.x}$. Thus, ω_0 and $\mathscr{E}_{HDW}$ are given by exactly the same coordinate formulas as $i^*_{\mathscr{P}}(\omega_1)$ and $\mathscr{E}_H^{\mathscr{P}}$. In particular, $\mathscr{E}_{HDW}$ itself covers $\mathscr{E}_{EL}$.

Finally, recall that the KdV equation is Hamiltonian, i.e., it can be presented in the form $u_t = A(\boldsymbol{E}(\mathscr{H}))$, where $\mathscr{H}$ is a top horizontal form in the infinite jet space of the

bundle $\mathbb{R}^2 \ni (x; u) \longmapsto x \in \mathbb{R}$, and A is a Hamiltonian $\mathscr{C}$-differential operator (see, for instance, [20]). Since Hamiltonian PDEs play a prominent role in the theory of integrable systems, it is worth to mention that such property (which is based on a 1+1, "covariance breaking" splitting of the space of independent variables (t, x)) is directly related with the present covariant Hamiltonian formalism as shown, for instance, in [29]. There the author provides a multisymplectic framework for the KdV equation by choosing, along the lines of [6], a "quasi-symmetric" Cartan form for the Lagrangian density (23). Such Cartan form is unique for a second order theory. Therefore, the formalism of [29] is actually equivalent to ours, in the special case of a second order theory.

Conclusions

In this paper, using the geometric theory of PDEs, we solved the long standing problem of finding a reasonably natural, higher order, field theoretic analogue of Hamiltonian mechanics of Lagrangian systems. By naturality we mean dependence on no structure other than the action functional. We achieved our goal in two steps. First we found a higher order, field theoretic analogue of the Skinner-Rusk mixed Lagrangian-Hamiltonian formalism [13, 14, 15] and, second, we showed that such theory projects naturally to a PD-Hamiltonian system on a smaller space. The obtained Hamiltonian field equations enjoy the following nice properties: 1) they are first order, 2) there is a canonical, non-local embedding of the Euler-Lagrange equations into them, and 3) for regular Lagrangian theories, they cover the Euler-Lagrange equations. Moreover, for regular Lagrangian theories, the coordinate expressions of the obtained field equations are nothing but the de Donder higher order field equations. This proves that our theory is truly the coordinate-free formulation of de Donder one [2].

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